Efficient CAD of discontinuities between elliptical and circular waveguides

Cristiano Tomassoni and Mauro Mongiardo

Istituto di Elettronica
Via G. Duranti, 93, 06131, Perugia, Italy.

Abstract— **Elliptical waveguides are currently finding several applications, since they provide improved flexibility with respect to circular waveguides and better manufacturability and higher Q with respect to rectangular waveguides. Effective CAD of components involving elliptical waveguides requires the efficient evaluation of the scattering parameters at the discontinuity occurring between elliptical and circular or rectangular waveguides.**
In this study we present analytical formulas for the efficient CAD of a junction between a circular and an elliptical waveguide of larger cross-section. With respect to current approaches, based on the numerical evaluation of the coupling integrals, the analytical formulas permits a significant reduction of computer time of more than one order of magnitude.
We have implemented the above formulas and results have been tested against published data and compared with those obtained by numerical integration: in all cases an almost perfect agreement is observed.

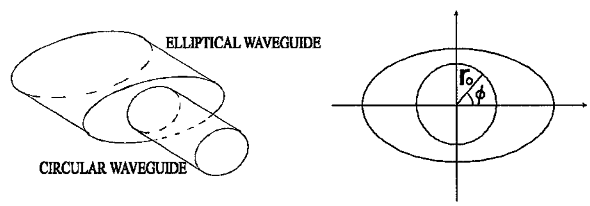

Fig. 1. Junction between an elliptical and a circular waveguide.

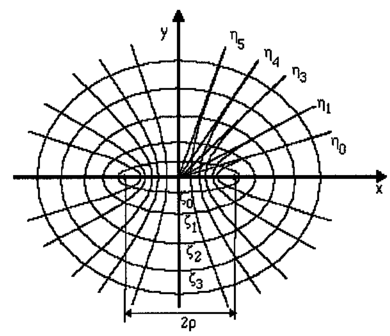

Fig. 2. Elliptical coordinate system of semifocal length ρ.

I. Introduction

Elliptical waveguides have recently found application in a variety of microwave components: their use has been proposed in dual mode filters [1], as low sensitivity irises, as matching sections between circular and rectangular waveguides, etc.. Waveguide discontinuities involving elliptical structures have received limited attention so far: the case of the junction between two confocal elliptical waveguides has been considered in [2], the general step discontinuity between two elliptical waveguides has been studied in [3], while the case of the junction between rectangular and elliptical waveguides has been considered in [4] and [5]; junctions between concentric elliptical and circular waveguides have been investigated in [5] for cross-section of the circular waveguide greater than that of the elliptical waveguide. Elliptical waveguides radiating into free space have been considered in [6] and, more recently in [7], where a transition between a rectangular waveguide and an elliptical one radiating into an half-space was studied. In all the above cases a modal solution of the discontinuity problem has been sought; the advantages of such a solution in terms of efficiency are well recognized. In [2], [5] the modal coupling coefficients have been obtained by analytical formulas; while in [3], [4] the evaluation of the coupling integrals has required numerical integration. Analytical solutions are extremely convenient since their use can substantially decrease the computation times; in several cases the above discontinuities need to be optimized with respect to the geometrical dimensions and it is therefore apparent that, whenever feasible, the use of analytical formulas is highly advisable. In this paper we consider the junction of a circular and an elliptical waveguide of larger cross-section (see Fig. 1). For this case not only we have found an analytical solution for the coupling coefficients, but the latter are provided by a single term expression which does not require computation of Mathieu functions. In the next section we present the theoretical evaluation of the coupling coefficients; in section III we compare the results of the proposed approach with published and reference data.

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II. Theory

A. Modal analysis in circular and elliptical waveguides

Modal analysis of the step discontinuity between an elliptical and a circular waveguide requires knowledge of the relative modal spectra. The normalized potentials of the modes of the circular waveguide, are well know and are here repeated in order to introduce notation (see also Fig. 1):

$$\psi'_{i,j}(\xi,\eta) = \frac{\sqrt{\frac{2}{\pi}} J_i(K'^{(c)}_{i,j} r) \begin{matrix}\cos\\ \sin\end{matrix} (i\phi)}{\epsilon_i K'^{(c)}_{i,j} r_o \left| J'_i(K'^{(c)}_{i,j} r_o)\right|}$$

$$\psi''_{i,j}(\xi,\eta) = \frac{\sqrt{\frac{2}{\pi}} J_i(K''^{(c)}_{i,j} r) \begin{matrix}\cos\\ \sin\end{matrix} (i\phi)}{\epsilon_i \sqrt{(K''^{(c)}_{i,j} r_o)^2 - i^2} \left| J_i(K'^{(c)}_{i,j} r_o)\right|} \quad (1)$$

where:

$$\epsilon_i = \begin{cases} \sqrt{2} & \text{if i=0} \\ 1 & \text{otherwise} \end{cases}$$

In the above formulas the symbol prime ($'$) is referred to TM modes while the symbol ($''$) is referred to TE modes; $K'^{(c)}_{i,j}$ and $K''^{(c)}_{i,j}$ are the cutoff wavenumber for $TM_{i,j}$ and $TE_{i,j}$, respectively, and $J'_i(x)$ is the first derivative of a Bessel function of order i. Moreover, from this point on, the superscript (c) denotes circular waveguides, while (e) is referred to elliptical waveguides.

Modal potentials for an elliptical waveguide are expressed as [8], [9]:

$$\psi'^{E/O}_{n,m}(\xi,\eta) = \begin{cases} NCe_n(\rho K'^{(e)E}_{n,m},\xi) ce_n(\rho K'^{(e)E}_{n,m},\eta) \Rightarrow TM^E_{n,m} \\ NSe_n(\rho K'^{(e)O}_{n,m},\xi) se_n(\rho K'^{(e)O}_{n,m},\eta) \Rightarrow TM^O_{n,m} \end{cases} \quad (2)$$

$$\psi''^{E/O}_{n,m}(\xi,\eta) = \begin{cases} NCe_n(\rho K''^{(e)E}_{n,m},\xi) ce_n(\rho K''^{(e)E}_{n,m},\eta) \Rightarrow TE^E_{n,m} \\ NSe_n(\rho K''^{(e)O}_{n,m},\xi) se_n(\rho K''^{(e)O}_{n,m},\eta) \Rightarrow TE^O_{n,m} \end{cases} \quad (3)$$

Here ξ and η are the radial and angular coordinates respectively of an elliptical coordinate system of the same semifocal length ρ of the elliptical waveguide (see Fig. 2). The functions ce_n ,Ce_n are respectively the *even Mathieu functions* and the *even modified Mathieu functions* of order n; similarly, the functions se_n, Se_n are the *odd Mathieu functions* and the *odd modified Mathieu functions* respectively. When the semifocal length of a elliptical coordinate system is fixed, the functions $Ce_n(\rho K^{(e)E}_{n,m},\xi)$, $Se_n(\rho K^{(e)O}_{n,m},\xi)$ depend on the cutoff wavenumber $K^{(e)E/O}_{n,m}$ and on the elliptical radial coordinate ξ; analogously, the functions $ce_n(\rho K^{(e)E}_{n,m},\xi)$, $se_n(\rho K^{(e)O}_{n,m},\xi)$ depend on the cutoff wavenumber $K^{(e)E/O}_{n,m}$ and on the elliptical angular coordinate η; N is the mode normalization constant. $TM^E_{n,m}$ and $TE^E_{n,m}$ are the *even modes*, while $TM^O_{n,m}$ and $TE^O_{n,m}$ are the *odd modes.*

With respect polarization, the meaning of even and odd is somewhat analogous to sin and cos in circular waveguide (1), while for the elliptical waveguide the odd modes are not the degenerate of even modes, because of the lack of symmetry of the ellipse with respect to the X and Y axis.

A useful expression of Mathieu functions is the following trigonometric expansion:

$$ce_{2\ell+np}(\rho K^{(e)E}_{n,m},\eta) = \sum_{r=0}^{\infty} A^{(2\ell+np)}_{2r+np} \cos(2r+np)\eta \quad (4)$$

$$se_{2\ell+np}(\rho K^{(e)O}_{n,m},\eta) = \sum_{r=0}^{\infty} B^{(2\ell+np)}_{2r+np} \sin(2r+np)\eta \quad (5)$$

In the above equations $np = 0, 1$, and the series expansion coefficient A and B (depending on cutoff wavenumber when ρ is fixed) are calculated as in [9], [10].

B. Coupling integrals

The generic coupling integral is defined as:

$$g_{p,q} = \int_{\mathcal{S}_c} \mathbf{e}^{(e)}_p \cdot \mathbf{e}^{(c)}_q ds \quad (6)$$

where $\mathcal{S}_c$ is the cross-section of the circular waveguide. p stands for a particular n,m,$'$ or $''$,E or O combination, while q stands for a particular i, j,$'$ and $''$, degenerate or non degenerate combination.

In order to evaluate *analytically* the above coupling integrals it is expedient to make use of the following expression [9], [10] which provide an expansion of the Mathieu functions in terms of the variables in the circular coordinate system:

$$Ce_{2\ell+np}(\rho K^{(e)E}_{n,m},\xi) ce_{2\ell+np}(\rho K^{(e)E}_{n,m},\eta) = \sqrt{\tfrac{\pi}{2}} ce_{2\ell+np}(\rho K''^{(e)E}_{n,m},0) \times \sum_{r=0}^{\infty} (-1)^{r+\ell} A^{(2\ell+np)}_{2r+np} \cos\{(2r+np)\phi\} J_{2r+np}(K''^{(e)E}_{n,m} r) \quad (7)$$

$$Se_{2\ell+np}(\rho K''^{(e)O}_{n,m},\xi) se_{2\ell+np}(\rho K''^{(e)O}_{n,m},\eta) = \sqrt{\tfrac{\pi}{2}} se'_{2\ell+np}(\rho K''^{(e)O}_{n,m},0) \times \sum_{r=0}^{\infty} (-1)^{r+\ell} B^{(2\ell+np)}_{2r+np} \sin\{(2r+np)\phi\} J_{2r+np}(K''^{(e)O}_{n,m} r) \quad (8)$$

where the coefficients A and B are those appearing in (4) and (5), while $n = 2\ell + np$. The point (ξ, η) in elliptical coordinates translates in a point (r, ϕ) in circular coordinates: accordingly, by using eq.

(8, 7) we can write the potential of elliptical waveguide in circular coordinates. We use the same normalization of [10] for Mathieu functions: accordingly, $ce_{2\ell+np}(\rho K^{(e)E}_{n,m},0)=1$ and $se'_{2\ell+np}(\rho K^{(e)O}_{n,m},0)=1$.

For deriving the analytical expressions of the coupling coefficient we insert eq. (7) and (8) in (2) and (3); then by substituting eq. (2, 3) and (1) into eq. (6), we obtain the followings formulas: for TM modes:

$$TM^{(e)E}_{2\ell+np,m} - TM^{(c)}_{i,j}\ ,\ TM^{(e)O}_{2\ell+np,m} - TM^{(c)d}_{i,j}$$
$$\textit{with (i-np) even}$$

$$g_{p,q} = \frac{(-1)^h \epsilon_i \pi N K^{(e)2}}{K^{(e)2} - K^{(c)2}} \left\{ \begin{array}{l} A^{2\ell+np}_i \\ B^{2\ell+np}_i \end{array} \right\} \frac{J'_i(K^{(c)}r_0)}{|J'_i(K^{(c)}r_0)|} J_i(K^{(e)}r_0)$$

For TE modes:

$$TE^{(e)E}_{2\ell+np,m} - TE^{(c)}_{i,j}\ ,\ TE^{(e)O}_{2\ell+np,m} - TE^{(c)d}_{i,j}$$
$$\textit{with (i-np) even}$$

$$g_{p,q} = \frac{(-1)^h \epsilon_i N K^{(c)^2}}{K^{(c)^2} - K^{(e)^2}} \left\{ \begin{array}{l} A^{2\ell+np}_i \\ B^{2\ell+np}_i \end{array} \right\} \frac{\pi K^{(e)} r_0}{\sqrt{(K^{(c)}r_0)^2 - i^2}} \times \frac{J_i(K^{(c)}r_0)}{|J_i(K^{(c)}r_0)|} J'_i(K^{(e)}r_0)$$

In the case of TM modes in elliptical waveguide and TE in circular waveguide:

$$TM^{(e)E}_{2\ell+np,m} - TE^{(c)d}_{i,j}\ ,\ TM^{(e)O}_{2\ell+np,m} - TE^{(c)}_{i,j}$$
$$\textit{with (i-np) even}$$

$$g_{p,q} = \left\{ \begin{array}{l} A^{2\ell+np}_i \\ -B^{2\ell+np}_i \end{array} \right\} \frac{(-1)^h N i \pi}{\sqrt{(K^{(c)}r_0)^2 - i^2}} \frac{J_i(K^{(c)}r_0)}{|J_i(K^{(c)}r_0)|} J_i(K^{(e)}r_0)$$

Finally, in all other cases:

$$\textit{OTHERWISE}$$

$$g_{p,q} = 0 \tag{9}$$

Where $h = \ell + \frac{i-np}{2}$. $K^{(e)}$ and $K^{(c)}$ are the cutoff wavenumber of the mode that we are considering in the elliptical and in the circular waveguide, respectively. The superscript (d) for circular modes denote a degenerate mode.

III. Results

Fig. 3 shows the reflection coefficient of a junction between a circular waveguide and an elliptical waveguide with very small eccentricity (nearly circular). We compare our results with data published in [2].

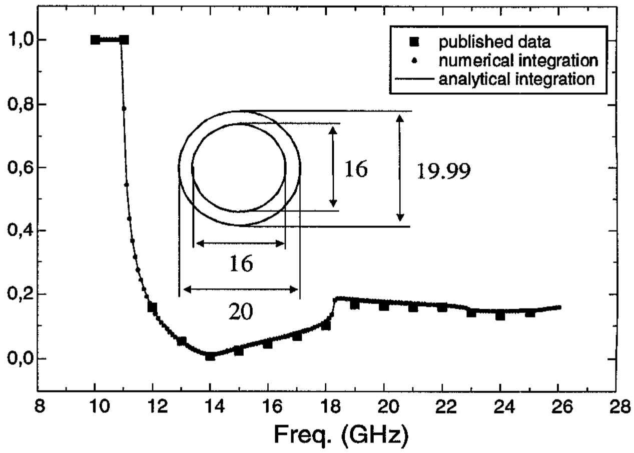

Fig. 3. Return loss for a junction between an elliptical waveguide with eccentricity 0.0316 and a circular waveguide. Comparison between [2] and results obtained by using the analytical and numerical evaluation of the coupling coefficients. Geometrical dimensions are expressed in mm.

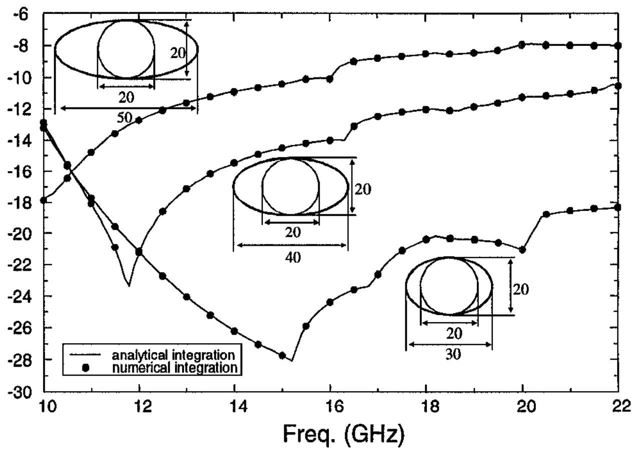

Fig. 4. Return Loss for three different discontinuities between elliptical and circular waveguides. Comparison between results obtained by using the analytical and numerical evaluation of the coupling coefficients. Geometrical dimensions are expressed in mm.

In the same graphic we have plotted a simulation obtained with a numerical solution of the coupling integral. It is noted that an excellent agreement is present between all the results presented; in particular the return loss obtained via analytical and numerical evaluation of the coupling coefficients coincides up to the third decimal figure. However, the accuracy of the numerical evaluation depends on a skilled choice of the number of integration points. As far as computation time are concerned it is noted that, for a single frequency point evaluation of the reflection coefficient, the code which employs analytical evaluation is about 40 times faster.

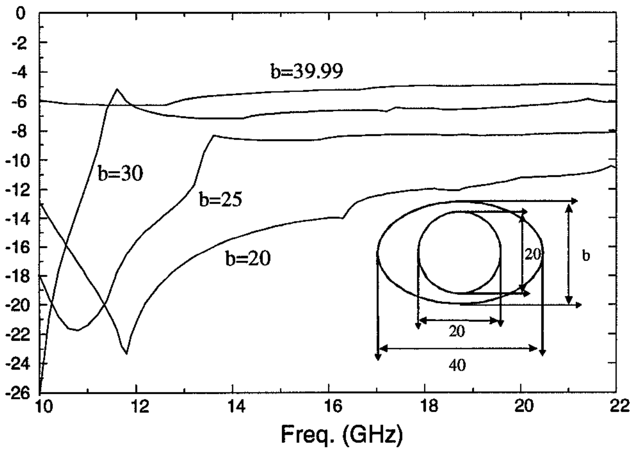

Fig. 5. Return Loss for four different discontinuities between elliptical and circular waveguides. In this graphic we have plotted four curves reporting the variation of the return loss with respect to variation of the minor axis of the elliptical waveguide. Geometrical dimensions are expressed in mm.

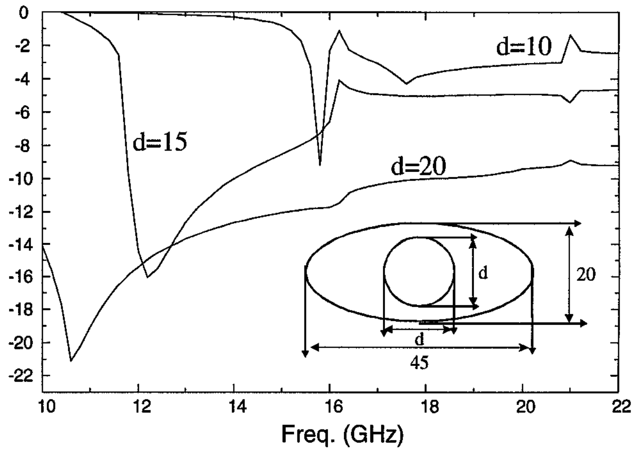

Fig. 6. Return Loss for three different discontinuities between elliptical and circular waveguides. In this graphic we have considered as fixed the ellipse geometrical dimensions and we have changed the diameter of the circular waveguide. Geometrical dimensions are expressed in mm.

In Fig. 4 we have plotted the reflection coefficient of a junction between a circular waveguide with radius 1 cm and an elliptical waveguide of semiminor axis of 1 cm and varying semimajor axis. We have compared these results with those obtained via numerical integration of the coupling integral. It can be seen that a very good agreement is always present. It is also apparent that the two approaches agree perfectly well (curves relative to our analytical and numerical integration are coincident). However it is noted that the computation time for a curve of 80 frequency point and a scattering matrix of 100x50 modes, is 260 seconds when using the analytical formulas, and about 2 hours when using numerical integration of coupling integral (on a PC Pentium 133 MHz).

In Fig. 5 we have plotted four curves reporting the variation of the return loss with respect to variation of the minor axis of the elliptical waveguide. It is apparent that the matching increases when the minor axis of the ellipse becomes equal to the diameter of the circular waveguide.

Finally, in Fig. 6 we have considered as fixed the ellipse geometrical dimensions (reported in the inset) and we have changed the diameter of the circular waveguide.

IV. Conclusion

We have presented an elegant analytical solution for the efficient CAD of junctions between a circular waveguide and an elliptical waveguide of larger cross-section. The coupling coefficients are evaluated by a single term expression which does not require any summation, thus avoiding possible problems of relative convergence. Computed results have been compared with published data and with other data obtained by numerical integration; in all cases an almost perfect agreement is observed. However, the code based on the analytical expression of the coupling coefficients has proven to be, for a typical case, about 40 times faster than the code based on the numerical evaluation of the coupling integrals.

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